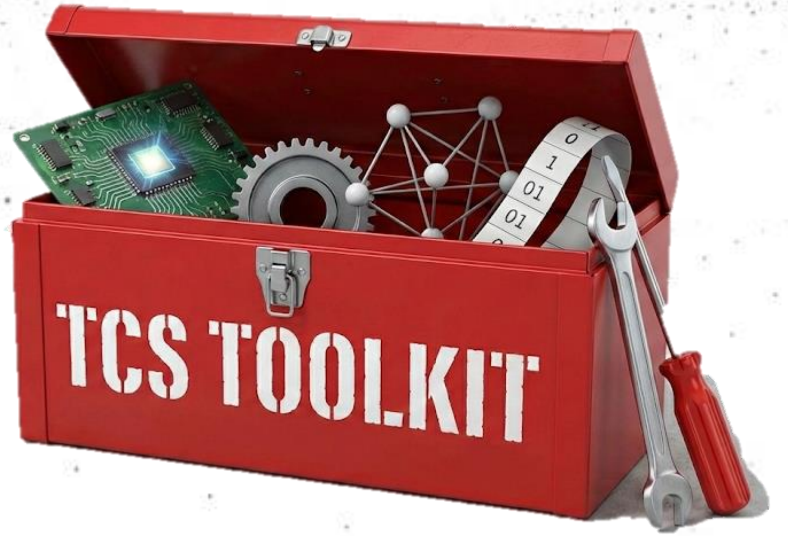
CS 58500 – Theoretical Computer Science Toolkit

Lecture 1 (01/21)

Mathematical Basics



https://ruizhezhang.com/course_spring_2026.html

Today’s Lecture

- Mathematical Inequalities
  - Taylor approximation
  - Jensen inequality

- Integration

- Stirling Approximation

Mathematical Inequalities: Taylor approximation

Taylor series

For any real or complex-valued function $f(x)$, if it is infinitely differentiable at $x = a$, then it has the following Taylor series expansion:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Examples:

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$
- $\sin(x) = x - \frac{x^3}{3} + \cdots$
- $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

Can we truncate at the k-th term and control the error?

Mathematical Inequalities: Taylor approximation

Lagrange Form of the Taylor Remainder Theorem

For any real or complex-valued function $f(x)$, under certain continuous condition around a, we have:

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + \underbrace{\frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1}}_{\text{Reminder } R_k(x)} \qquad \xi \in [x, a]$$

Examples:

- $\exp(\pm\epsilon) = 1 \pm \epsilon + \frac{\epsilon^2}{2} + \cdots + (-1)^k \frac{\epsilon^k}{k!} + \mathcal{O}(\epsilon^{k+1})$
- $\sin(\epsilon) = \epsilon + \mathcal{O}(\epsilon^3)$
- $\ln(1-\epsilon) = -\epsilon - \frac{\epsilon^2}{2} + \mathcal{O}(\epsilon^3)$

$$\sin(\epsilon) = 0 + \epsilon + 0 + R_2(\epsilon) = \epsilon - \frac{\cos(\xi)}{3!}\epsilon^3$$

- Since $|\cos(\xi)| \leq 1$, we have
$$\sin(\epsilon) = \epsilon + \mathcal{O}(\epsilon^3)$$
- Moreover, for small $\epsilon > 0$, $R_2(\epsilon) < 0$. Thus
$$\sin(\epsilon) < \epsilon$$

Mathematical Inequalities: Taylor approximation

Lagrange Form of the Taylor Remainder Theorem

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Examples:

- $\exp(\pm\epsilon) = 1 \pm \epsilon + \frac{\epsilon^2}{2} + \cdots + (-1)^k \frac{\epsilon^k}{k!} + \mathcal{O}\left(\epsilon^{k+1}\right)$
- $\sin(\epsilon) = \epsilon + \mathcal{O}(\epsilon^3)$
- $\ln(1-\epsilon) = -\epsilon - \frac{\epsilon^2}{2} + \mathcal{O}(\epsilon^3)$

Trick 1: based on the sign of $R_k(x)$, you can get lower or upper bounds:

- $\exp(-\epsilon) \geq 1 - \epsilon$
- $\ln(1-\epsilon) \leq -\epsilon$

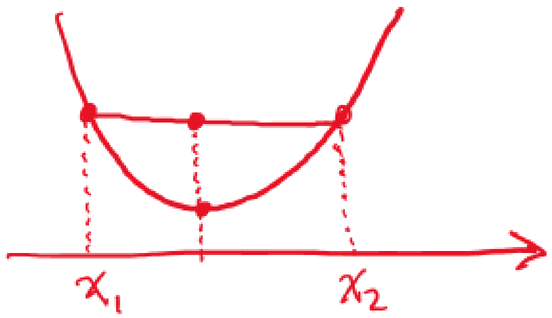
Trick 2: Desmos

Mathematical Basics: Convex function and Jensen's inequality

Convex function

Let f be a real-valued function. Then, f is convex in $[a, b]$ if one of the equivalent conditions holds:

- $\forall\, x \in [a, b], f^{(2)}(x) \geq 0$
- $\forall\, \lambda \in [0,1],\ x_1, x_2 \in [a, b], f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$



Jensen's inequality

For a convex f, $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$, "$=$" if and only if $x = y$

More generally, $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$ for any random variable

Mathematical Basics: Convex function and Jensen's inequality

Jensen's inequality

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \quad \forall \text{ convex } f$$

- AM-GM inequality:

$$\frac{a+b}{2} \geq \sqrt{ab}, \qquad \forall a, b > 0$$

  - $f(x) = -\ln x$

- Young's Inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \forall\, a, b > 0, \qquad p, q > 1\,, \frac{1}{p} + \frac{1}{q} = 1$$

  - $f(x) = -\ln(x)$

Mathematical Basics: Convex function and Jensen's inequality

Jensen's inequality

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \quad \forall \text{ convex } f$$

- Cauchy-Schwartz inequality:

$$|a_1 b_1 + \cdots + a_n b_n| \leq \left(\sum_i a_i^2\right)^{\frac{1}{2}} \left(\sum_i b_i^2\right)^{\frac{1}{2}}$$

$$\Leftrightarrow \quad \left(\sum_i p_i \frac{a_i}{b_i}\right)^2 \leq \sum_i p_i \left(\frac{a_i}{b_i}\right)^2, \qquad p_i \coloneqq \frac{b_i^2}{\sum_i b_i^2} \quad \forall i \in [n]$$

  - $f(x) = x^2$

- Hölder's inequality:

$$\sum_i |a_i b_i| \leq \left(\sum_i |a_i|^p\right)^{\frac{1}{p}} \left(\sum_i |b_i|^q\right)^{\frac{1}{q}}, \qquad \forall p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$$

Mathematical Basics: Convex function and Jensen’s inequality

Jensen’s inequality

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \quad \forall \text{ convex } f$$

- Minkowski’s inequality:

$$\left(\sum_i |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_i |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |b_i|^p\right)^{\frac{1}{p}} \quad \forall\, p \geq 1$$

  - $f(x) = x^p$

Trick 3: *Inequalities cheat sheet*

https://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf

- We first prove that

$$\sum_i (|a_i| + |b_i|)^p \le \left(\left(\sum_i |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_i |b_i|^p \right)^{\frac{1}{p}} \right)^p = \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p \right)^p$$

- Observe that

$$|a_i| + |b_i| = \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p \right) \left(\frac{\|\boldsymbol{a}\|_p}{\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p} \frac{|a_i|}{\|\boldsymbol{a}\|_p} + \frac{\|\boldsymbol{b}\|_p}{\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p} \frac{|b_i|}{\|\boldsymbol{b}\|_p} \right)$$

$$=: \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p \right) \left(\lambda \frac{|a_i|}{\|\boldsymbol{a}\|_p} + (1 - \lambda) \frac{|b_i|}{\|\boldsymbol{b}\|_p} \right)$$

- Since x^p is convex for $p \ge 1$, by Jensen's inequality,

$$(|a_i| + |b_i|)^p \le \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p \right)^p \left(\lambda \frac{|a_i|^p}{\|\boldsymbol{a}\|_p^p} + (1 - \lambda) \frac{|b_i|^p}{\|\boldsymbol{b}\|_p^p} \right)$$

- Summing over i:

$$\sum_i (|a_i| + |b_i|)^p \le \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p \right)^p \left(\lambda \sum_i \frac{|a_i|^p}{\|\boldsymbol{a}\|_p^p} + (1 - \lambda) \sum_i \frac{|b_i|^p}{\|\boldsymbol{b}\|_p^p} \right)$$

- Summing over i:

$$\sum_i (|a_i| + |b_i|)^p \le \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p\right)^p \left(\lambda \sum_i \frac{|a_i|^p}{\|\boldsymbol{a}\|_p^p} + (1-\lambda) \sum_i \frac{|b_i|^p}{\|\boldsymbol{b}\|_p^p}\right)$$
$$= \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p\right)^p \left(\lambda \frac{\|\boldsymbol{a}\|_p^p}{\|\boldsymbol{a}\|_p^p} + (1-\lambda) \frac{\|\boldsymbol{b}\|_p^p}{\|\boldsymbol{b}\|_p^p}\right)$$
$$= \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p\right)^p$$

- By triangle inequality,

$$|a_i + b_i| \le |a_i| + |b_i|$$

- Thus, we have

$$\sum_i |a_i + b_i|^p \le \sum_i (|a_i| + |b_i|)^p \le \left(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p\right)^p$$

$\blacksquare$

Today's Lecture

- Mathematical Inequalities
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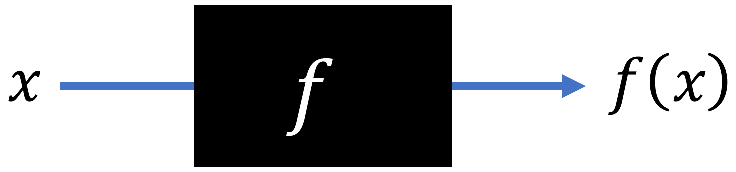
  - Jensen inequality
- Integration
- Stirling Approximation

Integration

Numerical integration

Given query access to $f:[a,b] \to \mathbb{R}$, how to estimate

$$\int_a^b f(x)\mathrm{d}x$$

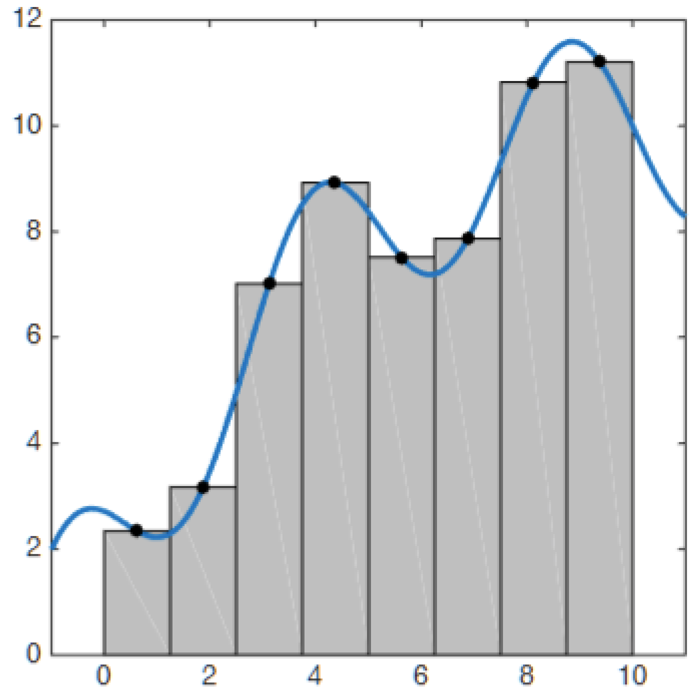


Meta algorithm: Quadrature

1. Choose a set of points and coefficients $\{(x_i, w_i)\}_{i\in[n]}$
2. Output

$$\sum_{i=1}^{n} w_i f(x_i)$$

Integration



Midpoint rule / Riemann sum

$$\int_0^1 f(x)\mathrm{d}x \approx \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i+0.5}{n}\right)$$

What is the approximation error?

Theorem (Midpoint rule error bound)**.**

Suppose that $|f''(x)| \leq M_2$ for $x \in [0,1]$. Then, we have

$$\left|\int_0^1 f(x)\mathrm{d}x - \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i+0.5}{n}\right)\right| \leq \frac{M_2}{24n^2}$$

Midpoint rule error bound proof

- Consider the first interval $\left[0, \frac{1}{n}\right]$. By Taylor's theorem,

$$f(x) = f\left(\frac{1}{2n}\right) + f'\left(\frac{1}{2n}\right)\left(x - \frac{1}{2n}\right) + f''(\xi_x)\left(x - \frac{1}{2n}\right)^2$$

- Integration:

$$\int_0^{\frac{1}{n}} f(x)\mathrm{d}x = \frac{1}{n} f\left(\frac{1}{2n}\right) + 0 + \int_0^{\frac{1}{n}} f''(\xi_x)\left(x - \frac{1}{2n}\right)^2 \mathrm{d}x$$

If $|f''(x)| \le M_2$ for $x \in [0,1]$

Midpoint rule error bound proof

- Consider the first interval $\left[0, \frac{1}{n}\right]$. By Taylor's theorem,

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- Integration:

$$\int_0^{\frac{1}{n}} f(x)\mathrm{d}x = \frac{1}{n} f\left(\frac{1}{2n}\right) + 0 + \int_0^{\frac{1}{n}} f''(\xi_x)\left(x - \frac{1}{2n}\right)^2 \mathrm{d}x$$

$$= \frac{1}{n} f\left(\frac{1}{2n}\right) \pm M_2 \int_0^{\frac{1}{n}} \left(x - \frac{1}{2n}\right)^2 \mathrm{d}x = \frac{1}{n} f\left(\frac{1}{2n}\right) \pm \frac{M_2}{24} n^{-3}$$

  if $|f''(x)| \leq M_2$ for $x \in [0,1]$

- Summing over n intervals together proves the theorem.

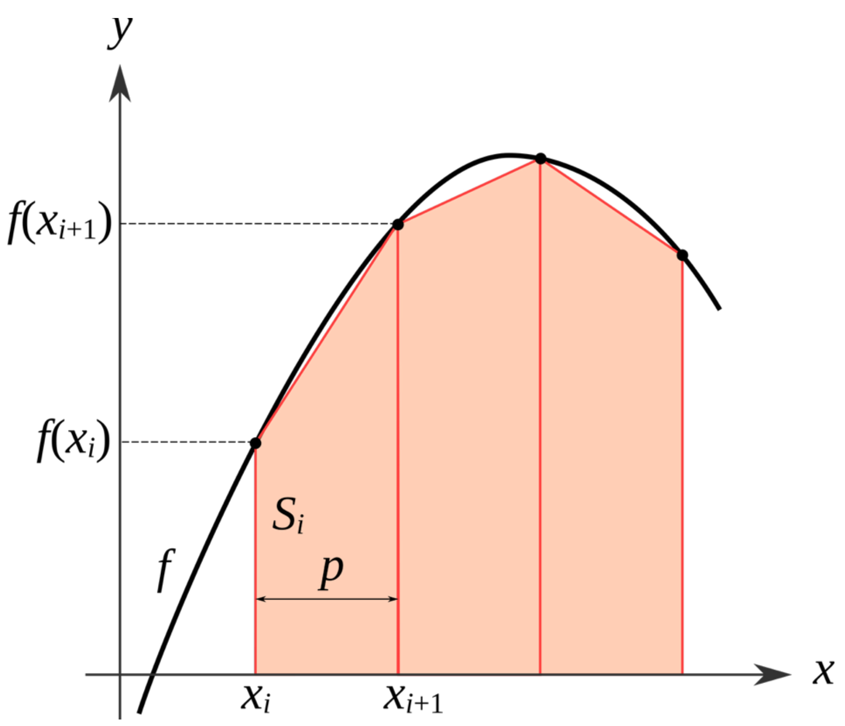
$\blacksquare$

Integration

Trapezoidal rule

$$\int_0^1 f(x)\mathrm{d}x \approx \sum_{i=0}^{n-1} \frac{1}{2}\left(f\left(\frac{i}{n}\right) + f\left(\frac{i+1}{n}\right)\right)\frac{1}{n}$$

$$= \frac{1}{2n}\left(f(0) + 2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + f(1)\right)$$



- Error bound: $\mathcal{O}(M_2 n^{-2})$

Integration

Interpolatory quadrature / Gaussian quadrature

$$\int_0^1 f(x)\mathrm{d}x \approx \int_0^1 p(x)\mathrm{d}x = \int_0^1 \sum_{i=0}^{n} f(x_i)\ell_i(x)\,\mathrm{d}x = \sum_{i=0}^{n} f(x_i) \underbrace{\int_0^1 \ell_i(x)\mathrm{d}x}_{w_i}$$

Lagrange interpolation

For any f, there is a unique degree-n polynomial p such that $f(x_i) = p(x_i)$ for any given $n+1$ points $x_0, \dots, x_n$. More specifically,

$$p(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)\,, \qquad \ell_i(x) \coloneqq \prod_{0 \le j \le n, j \ne i} \frac{x - x_j}{x_i - x_j}$$

Integration

Interpolatory quadrature / Gaussian quadrature

$$\int_0^1 f(x)\mathrm{d}x \approx \int_0^1 p(x)\mathrm{d}x = \int_0^1 \sum_{i=0}^{n} f(x_i)\ell_i(x)\,\mathrm{d}x = \sum_{i=0}^{n} f(x_i) \underbrace{\int_0^1 \ell_i(x)\mathrm{d}x}_{w_i}$$

- If f is a polynomial of degree $\leq n$, then there is no error!
- Suppose we have an interpolation scheme such that

$$\left| \int_0^1 f(x)\mathrm{d}x - \sum_{i=0}^{n} w_i f(x_i) \right| \leq C_n M_{n+1}, \qquad \text{where } M_{n+1} \coloneqq \max_{x\in[0,1]} \left| f^{(n+1)}(x) \right|$$

Then, we can cut the integral into k pieces and get the following

$$\left| \int_0^1 f(x)\mathrm{d}x - \frac{1}{k}\sum_{j=0}^{k-1}\sum_{i=0}^{n} w_i f\left(\frac{x_i + j}{k}\right) \right| \leq k \cdot C_n \frac{M_{n+1}}{k^{n+2}} = C_n \frac{M_{n+1}}{k^{n+1}}$$

(exp convergence!)

Apply chain rule $n+1$ times

Integration

Interpolatory quadrature / Gaussian quadrature

$$\left|\int_0^1 f(x)\mathrm{d}x - \sum_{i=0}^{n} w_i f(x_i)\right| \le C_n M_{n+1}, \qquad \text{where } M_{n+1} \coloneqq \max_{x\in[0,1]} \left|f^{(n+1)}(x)\right|$$

Lemma. Given any $n+1$ times differentiable function f. Let $p_n(x)$ be an n degree polynomial such that $p_n(x_i) = f(x_i)$ for $i = 0,1,\dots,n$. Then, for any $x \in [0,1]$, we have

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x - x_i) \quad \text{for some } \xi_x \in [0,1]$$

- This lemma implies that

$$\left|\int_0^1 f(x)\mathrm{d}x - \sum_{i=0}^{n} w_i f(x_i)\right| \le \left|\int_0^1 \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)\,\mathrm{d}x\right| \le \frac{M_{n+1}}{(n+1)!}\cdot \max_{x\in[0,1]}\left|\prod_{i=0}^{n}(x - x_i)\right|$$

Integration

Lemma. Given any $n+1$ times differentiable function f. Let $p_n(x)$ be an n degree polynomial such that $p_n(x_i) = f(x_i)$ for $i = 0,1,\ldots,n$. Then, for any $x \in [0,1]$, we have

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Proof.

- Let $\omega(t) \coloneqq \prod_{i=0}^{n}(t - x_i)$ and consider the function

$$F(t) \coloneqq f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\omega(x)}\omega(t)$$

- $F(x_i) = 0$ for $i = 0,1,\ldots,n$ and $F(x) = 0$ (check by yourself)
- $F(t)$ has $n+2$ roots in $[0,1]$, by applying Rolle's Theorem repeatedly, we get that there exists $\xi \in [0,1]$ such that $F^{(n+1)}(\xi) = 0$

$$F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - 0 - \frac{f(x) - p_n(x)}{\omega(x)} = 0$$

■

Integration

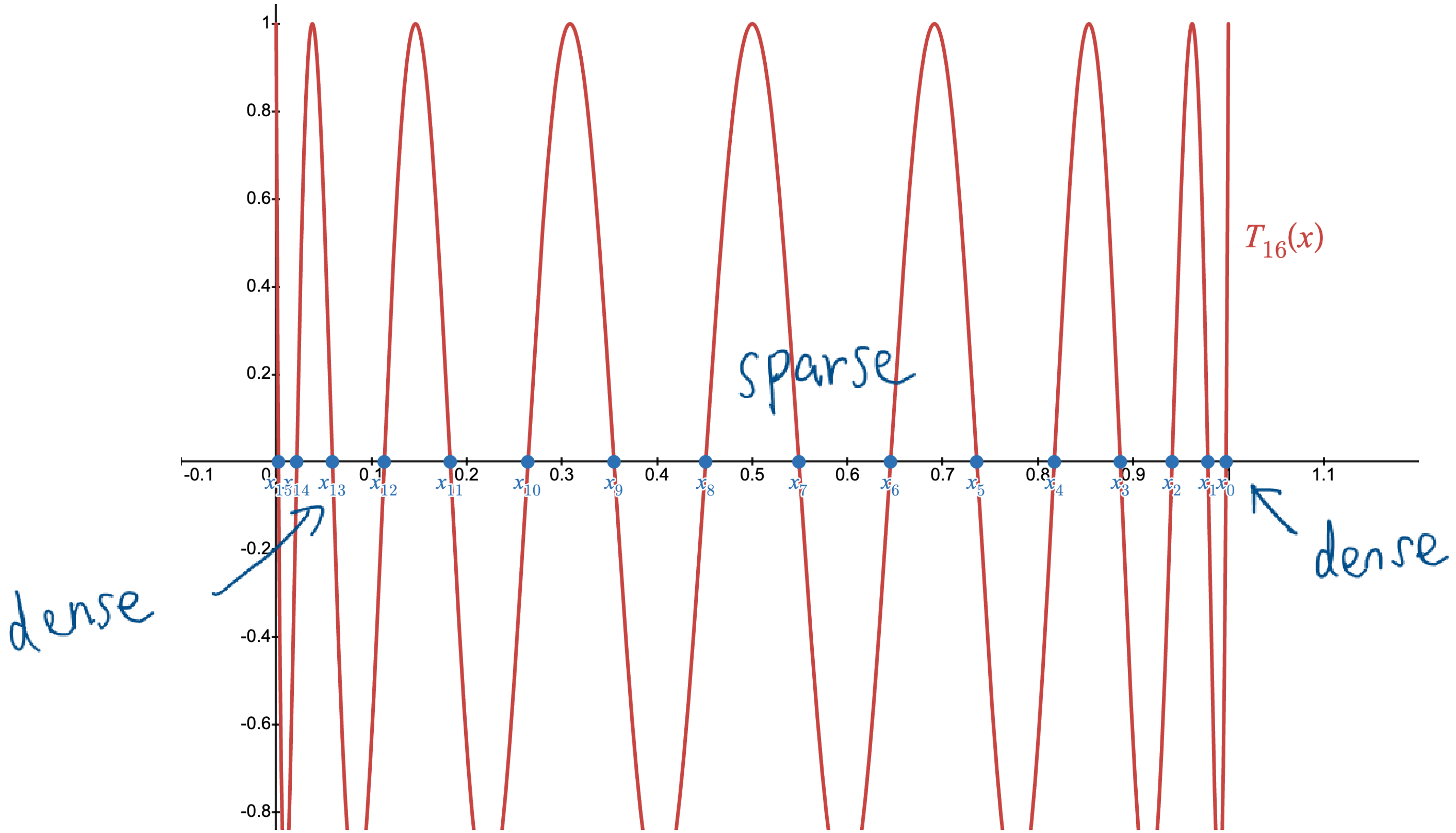
Interpolatory quadrature / Gaussian quadrature

$$\left|\int_0^1 f(x)\mathrm{d}x - \sum_{i=0}^{n} w_i f(x_i)\right| \le C_n M_{n+1}, \qquad \text{where } M_{n+1} \coloneqq \max_{x\in[0,1]} \left|f^{(n+1)}(x)\right|$$

- How to choose $x_0, x_1, \dots, x_n \in [0,1]$ such that $\max_{x\in[0,1]} \left|\prod_{i=0}^{n}(x - x_i)\right|$ is small?
- This quantity is minimized by Chebyshev nodes:

$$x_i \coloneqq \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2i+1}{2n+2}\pi\right) \quad \forall i = 0,1,\dots,n$$

  - $\{x_i\}$ are the roots of the Chebyshev polynomial $T_{n+1}(2x-1)$



https://www.desmos.com/calculator/dnyqzmwkei

Integration

Interpolatory quadrature / Gaussian quadrature

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- How to choose $x_0, x_1, \dots, x_n \in [0,1]$ such that $\max_{x\in[0,1]} \left|\prod_{i=0}^{n}(x - x_i)\right|$ is small?
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$$x_i \coloneqq \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2i+1}{2n+2}\pi\right) \quad \forall i = 0,1,\dots,n$$

- We have that

$$\left|\int_0^1 f(x)\mathrm{d}x - \sum_{i=0}^{n} w_i f(x_i)\right| \leq \frac{M_{n+1}}{(n+1)!\, 2^{2n+1}}$$

Integration

Exponentially Convergent Trapezoidal Rule

- If f is sufficiently smooth and periodic, then the trapezoidal rule / midpoint rule has an exponentially convergent error bound.
- Exponential convergence also holds for some "peak-like" functions integrated over the real line, e.g.,

$$\left|\underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x}_{\sqrt{\pi}} - h \sum_{j=-\infty}^{\infty} e^{-(jh)^2}\right| = \mathcal{O}\left(e^{-\pi^2/h^2}\right)$$

Exponentially Convergent Trapezoidal Rule

- Let $I_h := h\sum_{j=-\infty}^{\infty} e^{-(jh)^2}$ and $I = \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x$
- Consider the Fourier transform of $f(x) = e^{-x^2}$:

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathbf{i}\xi x} e^{-x^2}\mathrm{d}x = \frac{1}{2\sqrt{\pi}} e^{-\xi^2/4}$$

- Observe that $I = 2\pi\hat{f}(0) = \sqrt{\pi}$
- Another tool we use is the Poisson Summation Formula, which connects discrete sum I_h to the continuous Fourier transform $\hat{f}$:

$$I_h = 2\pi\sum_{j=-\infty}^{\infty} \hat{f}\left(\frac{2\pi j}{h}\right) = \sqrt{\pi}\sum_{j=-\infty}^{\infty} e^{-\pi^2 j^2/h^2}$$

$$|I - I_h| = 2\sqrt{\pi}\sum_{j=1}^{\infty} e^{-\pi^2 j^2/h^2}$$

Exponentially Convergent Trapezoidal Rule

$$I = 2\pi \hat{f}(0)$$

$$I_h = 2\pi \sum_{j=-\infty}^{\infty} \hat{f}\left(\frac{2\pi j}{h}\right) = \sqrt{\pi} \sum_{j=-\infty}^{\infty} e^{-\pi^2 j^2/h^2}$$

- Thus, we have

$$|I - I_h| = 2\pi \sum_{j\neq 0} \hat{f}\left(\frac{2\pi j}{h}\right) = 2\sqrt{\pi} \sum_{j=1}^{\infty} e^{-\pi^2 j^2/h^2}$$

$$\leq 2\sqrt{\pi} \sum_{j=1}^{\infty} \left(e^{-\pi^2/h^2}\right)^j = 2\sqrt{\pi} e^{-\pi^2/h^2} \frac{1}{1 - e^{-\pi^2/h^2}} = \mathcal{O}\left(e^{-\pi^2/h^2}\right)$$

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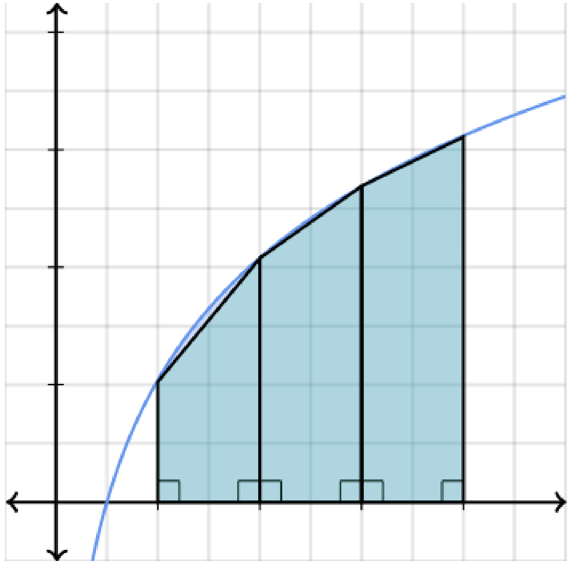
Stirling Approximation



$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

- $\ln(n!) = \sum_{i=1}^{n} \ln(i)$
- Trapezoidal rule:

concave
increasing

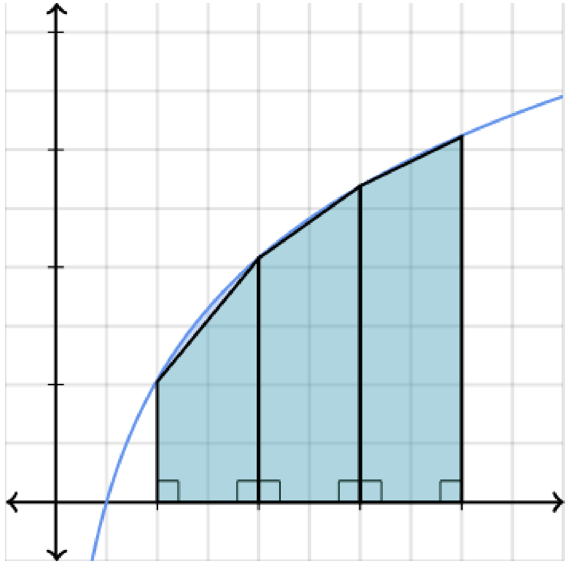
$$\int_1^n \ln(x)\,\mathrm{d}x \overset{\geq}{\approx} \frac{1}{2}(\ln(1) + \ln(n)) + \sum_{i=2}^{n-1} \ln(i) = \ln(n!) - \frac{1}{2}\ln(n)$$

- Evaluate the integral and simplify the terms:

$$\int_1^n \ln(x)\,\mathrm{d}x = x\ln(x) - x\,\Big|_1^n = n\ln(n) - n + 1$$

$$\ln(n!) \approx \left(n + \frac{1}{2}\right)\ln(n) - n + 1 \qquad \Longrightarrow \qquad n! \underset{\leq}{\approx} e\sqrt{n}\left(\frac{n}{e}\right)^n$$

Stirling Approximation



- Trapezoidal rule:

$$\int_1^n \ln(x)\,\mathrm{d}x = \ln(n!) - \frac{1}{2}\ln(n) + E_n, \qquad (E_n > 0)$$

- Evaluate the integral and simplify the terms:

$$\ln(n!) = \left(n + \frac{1}{2}\right)\ln(n) - n + 1 - E_n \qquad \Longrightarrow \qquad n! = e^{1-E_n}\sqrt{n}\left(\frac{n}{e}\right)^n$$

- For each segment $[i, i+1]$,

$$\int_i^{i+1} \ln(x)\,\mathrm{d}x - \frac{1}{2}(\ln(i) + \ln(i+1)) = -\frac{(\ln(x))''|_{x=\xi_i}}{12} = \frac{1}{12\xi_i^2}, \qquad \xi_k \in (i, i+1)$$

- Summing together,

$$E_n = \sum_{i=1}^{n-1} \frac{1}{12\xi_i^2} < \sum_{i=1}^{n-1} \frac{1}{12i^2} = \mathcal{O}(1), \qquad E_n > \sum_{i=1}^{n-1} \frac{1}{12(i+1)^2} = \Omega(1) \quad \Longrightarrow \quad E_n = C + o(1)$$

Stirling Approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)$$

Asymptotic estimation of binormal coefficient

- Basic version:

$$\frac{n^k}{k^k} \le \binom{n}{k} < \left(\frac{en}{k}\right)^k$$

- Entropy version:

$$\frac{2^{nH(p)}}{\sqrt{8p(1-p)n}} \le \binom{n}{pn} \le \frac{2^{nH(p)}}{\sqrt{2\pi p(1-p)n}}$$

where $H(p) \coloneqq -p\log_2(p) - (1-p)\log_2(1-p)$ for any $p \in (0,1)$

Binormal Coefficient Estimation

$$\frac{2^{nH(p)}}{\sqrt{8p(1-p)n}} \leq \binom{n}{pn} \leq \frac{2^{nH(p)}}{\sqrt{2\pi p(1-p)n}}$$

- By Stirling approximation,

$$\binom{n}{pn} \approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi pn}\left(\frac{pn}{e}\right)^{pn} \cdot \sqrt{2\pi(1-p)n}\left(\frac{(1-p)n}{e}\right)^{(1-p)n}}$$

$$= \frac{1}{\sqrt{2\pi p(1-p)n}} \cdot e^{pn+(1-p)n-n} \cdot 2^{n\left(\log n - p\log(pn) - (1-p)\log\left((1-p)n\right)\right)}$$

$$= \frac{1}{\sqrt{2\pi p(1-p)n}} \cdot 2^{n(-p\log p - (1-p)\log(1-p))} = \frac{2^{nH(p)}}{\sqrt{2\pi p(1-p)n}}$$